Polytechnic Journal. 2022. 12(2): 6-16
ISSN:
http://journals.epu.edu.iq/index.php/polytechnic

RESEARCH ARTICLE

SEE Transform Technique for Solving System of Linear Volterra Integro-Differential Equations of the Second Kind

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Received: 13 Apr 2022
Accepted: 10 May 2022
Published: 1 February 2023
DOI
10.25156/ptj.v12n2y2022.pp6-16

ABSTRACT

There are many integral transforms that are widely used to solve numerous real-life, science, and engineering problems. In this article, we present SEE transform for determining the solution of the system of linear Volterra integro-differential equations of the second kind. Some applications have been given and solved by using the SEE transform for illustrating the applicability of the SEE transform. Results of the applications assert that the SEE transform is very effective for obtaining the exact solution of this equation.

Key Words: Volterra Integro-Differential Equation, SEE Transform, Convolution, Inverse SEE Transform

INTRODUCTION

The system of linear Volterra integro-differential equations of the second kind (S-LVIDE-SK) is given by Aggarwal and Kumar (2021)

$$\begin{cases} v_1^{(m)}(s) = h_1(s) + \int_0^s K_{11}(s-t)\, v_1(t)dt + \int_0^s K_{12}(s-t)v_2(t)dt + \cdots + \int_0^s K_{1n}(s-t)v_n(t)dt \\ v_2^{(m)}(s) = h_2(s) + \int_0^s K_{21}(s-t)\, v_1(t)dt + \int_0^s K_{22}(s-t)v_2(t)dt + \cdots + \int_0^s K_{2n}(s-t)v_n(t)dt \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n^{(m)}(s) = h_n(s) + \int_0^s K_{n1}(s-t)\, v_1(t)dt + \int_0^s K_{n2}(s-t)v_2(t)dt + \cdots + \int_0^s K_{nn}(s-t)v_n(t)dt \end{cases}$$

With

$$\begin{cases} v_1^{(l)}(0) = a_{1l}, & l = 0,1,2,\dots,m-1; \\ v_2^{(l)}(0) = a_{2l}, & l = 0,1,2,\dots,m-1; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n^{(l)}(0) = a_{nl}, & l = 0,1,2,\dots,m-1; \end{cases}$$

Where the unknown functions $v_1(t), v_2(t) \dots, v_n(t)$ which will be determined, appear only inside the integral sign whilst the derivatives of $v_1(t), v_2(t) \dots, v_n(t)$ mostly occur outside the integral sign. The Kernels $K_{ji}(s,t)$, and the function $h_j(s)$ for $j,i = 1, 2, \dots, n$ are given real-valued functions. There are many phenomena and processes in different areas of engineering and applied science where integro-ordinary differential equation plays an important role like in circuit analysis, glass forming process, wave propagation, nuclear reactors, nano-hydrodynamics, biological population, visco-elasticity, optimal control system, geophysics, population genetics, radiation optimization, hereditary phenomena in biology and physics, magnetism and electricity, medicine, kinetic theory of gases, communication theory, quantum mechanics.

The SEE transform of the function W(t) is defined as Ajel Mansour et al. (2021):

$$S\{W(t)\} = \frac{1}{q^n}\int_0^\infty W(t)e^{-qt}dt = G(q), \qquad n \in \mathbb{Z},$$

$$t \geq 0,\ \ell_1 \leq q \leq \ell_2 \qquad (1)$$

Where S is SEE transform operator.

The SEE transform of the function W(t) for $t \geq 0$ exist if W(t) is piecewise continuous and of exponential order. These conditions are only sufficient conditions for existence of SEE transform of the functionW(t).

In recently years some researchers for solving system of Fredholm and Volterra integro-differential equations have used several techniques.

Aggarwal and Kumar (2021) used Laplace-Carson transform for solving (S-LVIDE-SK). Jalal et al. (2019) solved the (S-LVIDE-SK) by modified decomposition method. Rabiei et al. (2019) they considered a third order General Linear Method for finding the numerical solution of Volterra integro-differential equation . Hasan and Suleiman (2018) by applying Linear Programming problem demonstrated the numerical solution of mixed Voltera-Fredholm integral equations. Hassan et al. (2017) applied Aitken method for solving Volterra-Fredholm integral equations of the second kind with Homotopy perturbation technique. Bakodah et al. (2017) showed an efficient modification of Adomian decomposition method to solve the non-linear system of Fredholm and Volterra integro-differential equations. Chandra Guru Sekar et al. (2017) they studied the Single Term Walsh Series approach for solving the (S-LVIDE-SK) and their numerical solutions. Singh and Wazwaz (2016) they proposed a reliable technique based on Adomian decomposition method (ADM) for the numerical solution of fourth-order boundary value problems for Volterra integro-differential equations. Hassan (2011) used power functions to determine the solution of system of Fredholm integral equation of the second kind.

If we concentrate on the SEE transform, Mansou et al. (2021)showed the usefulness of the SEE integral transform of Bessel’s functions to determine the integral which includes Bessel’s functions.

The main purpose of this paper is to determine the solution of the system of linear Volterra integro-differential equations of the second kind by applying SEE transform.

SEE TRANSFORM TO SOLVE SYSTEM OF LINEAR VOLTERA INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND

The general (S-LVIDE-SK) is given by Wazwaz (2011),

$$\left.\begin{array}{l}
v_1^{(m)}(s) = h_1(s) + \left\{\begin{array}{c}\int_0^s K_{11}(s-t)\, v_1(t)dt + \int_0^s K_{12}(s-t)v_2(t)dt \\ + \cdots + \int_0^s K_{1n}(s-t)v_n(t)dt\end{array}\right\} \\
v_2^{(m)}(s) = h_2(s) + \left\{\begin{array}{c}\int_0^s K_{21}(s-t)v_1(t)dt + \int_0^s K_{22}(s-t)v_2(t)dt \\ + \cdots + \int_0^s K_{2n}(s-t)v_n(t)dt\end{array}\right\} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\
v_n^{(m)}(s) = h_n(s) + \left\{\begin{array}{c}\int_0^s K_{n1}(s-t)v_1(t)dt + \int_0^s K_{n2}(s-t)v_2(t)dt \\ + \cdots + \int_0^s K_{nn}(s-t)v_n(t)dt\end{array}\right\}
\end{array}\right] \qquad (2)$$

$$\text{with} \left\{\begin{array}{l}
v_1^{(l)}(0) = a_{1l},\ l = 0,1,2,\dots,m-1; \\
v_2^{(l)}(0) = a_{2l},\ l = 0,1,2,\dots,m-1; \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
v_n^{(l)}(0) = a_{nl},\ l = 0,1,2,\dots,m-1;
\end{array}\right\} \qquad (3)$$

Taking ESS transform's operator on system (2) and using convolution theorem of ESS transform, we get

$$\left.\begin{array}{l} S\left\{v_1^{(\mathrm{m})}(\mathrm{s})\right\} = S\{h_1(\mathrm{s})\} + \begin{bmatrix} q^n S\{K_{11}(\mathrm{s})\}S\{v_1(\mathrm{s})\} + q^n S\{K_{12}(\mathrm{s})\}S\{v_2(\mathrm{s})\} \\ + \cdots + q^n S\{K_{1n}(\mathrm{s})\}S\{v_n(\mathrm{s})\} \end{bmatrix} \\ S\left\{v_2^{(\mathrm{m})}(\mathrm{s})\right\} = S\{h_2(\mathrm{s})\} + \begin{bmatrix} q^n S\{K_{21}(\mathrm{s})\}S\{v_1(\mathrm{s})\} + q^n S\{K_{22}(\mathrm{s})\}S\{v_2(\mathrm{s})\} \\ + \cdots + q^n S\{K_{2n}(\mathrm{s})\}S\{v_n(\mathrm{s})\} \end{bmatrix} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ S\left\{v_n^{(\mathrm{m})}(\mathrm{s})\right\} = S\{h_n(\mathrm{s})\} + \begin{bmatrix} q^n S\{K_{n1}(\mathrm{s})\}S\{v_1(\mathrm{s})\} + q^n S\{K_{n2}(\mathrm{s})\}S\{v_2(\mathrm{s})\} \\ + \cdots + q^n S\{K_{nn}(\mathrm{s})\}S\{v_n(\mathrm{s})\} \end{bmatrix} \end{array}\right] \tag{4}$$

Using the property “ESS transforms of derivatives” on system (4), we get

$$\left.\begin{array}{l} \begin{Bmatrix} -\frac{v_1^{(m-1)}(0)}{q^n} \\ -\frac{v_1^{(m-2)}(0)}{q^{n-1}} \\ -\cdots-\frac{v_1(0)}{q^{n-m+1}} \\ +q^m S\{v_1(s)\} \end{Bmatrix} = S\{h_1(s)\} + \begin{bmatrix} q^n S\{K_{11}(s)\}S\{v_1(s)\} + q^n S\{K_{12}(s)\}S\{v_2(s)\} \\ + \cdots + q^n S\{K_{1n}(s)\}\, S\{v_n(s)\} \end{bmatrix} \\ \begin{Bmatrix} -\frac{v_2^{(m-1)}(0)}{q^n} \\ -\frac{v_2^{(m-2)}(0)}{q^{n-1}} \\ -\cdots-\frac{v_2(0)}{q^{n-m+1}} \\ +q^m S\{v_2(s)\} \end{Bmatrix} = S\{h_2(s)\} + \begin{bmatrix} q^n S\{K_{21}(s)\}S\{v_1(s)\} + q^n S\{K_{22}(s)\}S\{v_2(s)\} \\ + \cdots + q^n S\{K_{2n}(s)\}\, S\{v_n(s)\} \end{bmatrix} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \begin{Bmatrix} -\frac{v_n^{(m-1)}(0)}{q^n} \\ -\frac{v_n^{(m-2)}(0)}{q^{n-1}} \\ -\cdots-\frac{v_n(0)}{q^{n-m+1}} \\ +q^m S\{v_n(s)\} \end{Bmatrix} = S\{h_n(s)\} + \begin{bmatrix} q^n S\{K_{n1}(s)\}S\{v_1(s)\} + q^n S\{K_{n2}(s)\}S\{v_2(s)\} \\ + \cdots + q^n S\{K_{nn}(s)\}\, S\{v_n(s)\} \end{bmatrix} \end{array}\right] \tag{5}$$

Substituting equation (3) in system (5), we obtain

$$\left[\begin{array}{l}\left\{\begin{array}{c}-\frac{a_{1(m-1)}}{q^n}\\ -\frac{a_{1(m-2)}}{q^{n-1}}\\ -\cdots-\frac{a_{10}}{q^{n-m+1}}\\ +q^m S\{v_1(s)\}\end{array}\right\} = S\{h_1(s)\} + \left[\begin{array}{c}S\{K_{11}(s)\}S\{v_1(s)\}\\ +S\{K_{12}(s)\}S\{v_2(s)\}\\ +\cdots+S\{K_{1n}(s)\}S\{v_n(s)\}\end{array}\right]\\ \left\{\begin{array}{c}-\frac{a_{2(m-1)}}{q^n}\\ -\frac{a_{2(m-2)}}{q^{n-1}}\\ -\cdots-\frac{a_{20}}{q^{n-m+1}}\\ +q^m S\{v_2(s)\}\end{array}\right\} = S\{h_2(s)\} + \left[\begin{array}{c}S\{K_{21}(s)\}S\{v_1(s)\}\\ +S\{K_{22}(s)\}S\{v_2(s)\}\\ +\cdots+S\{K_{2n}(s)\}S\{v_n(s)\}\end{array}\right]\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \left\{\begin{array}{c}-\frac{a_{n(m-1)}}{q^n}\\ -\frac{a_{n(m-2)}}{q^{n-1}}\\ -\cdots-\frac{a_{n0}}{q^{n-m+1}}\\ +q^m S\{v_n(s)\}\end{array}\right\} = S\{h_n(s)\} + \left[\begin{array}{c}S\{K_{n1}(s)\}S\{v_1(s)\}\\ +S\{K_{n2}(s)\}S\{v_2(s)\}\\ +\cdots+S\{K_{nn}(s)\}S\{v_n(s)\}\end{array}\right]\end{array}\right] \tag{6}$$

After simplification system (6), we get

$$\left[\begin{array}{l}\left[\begin{array}{c}(q^m - S\{K_{11}(s)\})S\{v_1(s)\} - S\{K_{12}(s)\}S\{v_2(s)\}\\ -\cdots-S\{K_{1n}(s)\}S\{v_n(s)\}\end{array}\right] = \left[\begin{array}{c}S\{h_1(s)\}\\ +\frac{a_{1(m-1)}}{q^n}\\ +\frac{a_{1(m-2)}}{q^{n-1}}\\ +\cdots+\frac{a_{10}}{q^{n-m+1}}\end{array}\right]\\ \left[\begin{array}{c}-S\{K_{21}(s)\}\, S\{v_1(s)\} + (q^m - S\{K_{22}(s)\})S\{v_2(s)\}\\ -\cdots-S\{K_{2n}(s)\}\, S\{v_n(s)\}\end{array}\right] = \left[\begin{array}{c}S\{h_2(s)\}\\ +\frac{a_{2(m-1)}}{q^n}\\ +\frac{a_{2(m-2)}}{q^{n-1}}\\ +\cdots+\frac{a_{20}}{q^{n-m+1}}\end{array}\right]\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \left[\begin{array}{c}-S\{K_{n1}(s)\}\, S\{v_1(s)\} - S\{K_{n2}(s)\}\, S\{v_2(s)\} - \cdots\\ +(q^m - S\{K_{nn}(s)\})S\{v_n(s)\}\end{array}\right] = \left[\begin{array}{c}S\{h_n(s)\}\\ +\frac{a_{n(m-1)}}{q^n}\\ +\frac{a_{n(m-2)}}{q^{n-1}}\\ +\cdots+\frac{a_{n0}}{q^{n-m+1}}\end{array}\right]\end{array}\right] \tag{7}$$

The solution of system (7) is given as

$$S\{v_1(s)\} = \frac{\begin{vmatrix} \left\{\begin{matrix} S\{h_1(s)\} \\ +\frac{a_{1(m-1)}}{q^n} \\ +\frac{a_{1(m-2)}}{q^{n-1}} \\ +\cdots+\frac{a_{10}}{q^{n-m+1}} \end{matrix}\right\} & -S\{K_{12}(s)\} & \ldots & -S\{K_{1n}(s)\} \\ \left\{\begin{matrix} S\{h_2(s)\} \\ +\frac{a_{2(m-1)}}{q^n} \\ +\frac{a_{2(m-2)}}{q^{n-1}} \\ +\cdots+\frac{a_{20}}{q^{n-m+1}} \end{matrix}\right\} & (q^m - S\{K_{22}(s)\}) & \ldots & -S\{K_{2n}(s)\} \\ \ldots & \ldots & \ldots & \ldots \\ \left\{\begin{matrix} S\{h_n(s)\} \\ +\frac{a_{n(m-1)}}{q^n} \\ +\frac{a_{n(m-2)}}{q^{n-1}} \\ +\cdots+\frac{a_{n0}}{q^{n-m+1}} \end{matrix}\right\} & -S\{K_{n2}(s)\} & \ldots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \ldots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \ldots & -S\{K_{2n}(s)\} \\ \ldots & \ldots & \ldots & \ldots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \ldots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}$$

$$S\{v_2(s)\} = \frac{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & \left\{\begin{matrix} S\{h_1(s)\} \\ +\frac{a_{1(m-1)}}{q^n} \\ +\frac{a_{1(m-2)}}{q^{n-1}} \\ +\cdots+\frac{a_{10}}{q^{n-m+1}} \end{matrix}\right\} & \ldots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & \left\{\begin{matrix} S\{h_2(s)\} \\ +\frac{a_{2(m-1)}}{q^n} \\ +\frac{a_{2(m-2)}}{q^{n-1}} \\ +\cdots+\frac{a_{20}}{q^{n-m+1}} \end{matrix}\right\} & \ldots & -S\{K_{2n}(s)\} \\ \ldots & \ldots & \ldots & \ldots \\ -S\{K_{n1}(s)\} & \left\{\begin{matrix} S\{h_n(s)\} \\ +\frac{a_{n(m-1)}}{q^n} \\ +\frac{a_{n(m-2)}}{q^{n-1}} \\ +\cdots+\frac{a_{n0}}{q^{n-m+1}} \end{matrix}\right\} & \ldots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \ldots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \ldots & -S\{K_{2n}(s)\} \\ \ldots & \ldots & \ldots & \ldots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \ldots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}$$

....

$$S\{v_n(s)\} = \frac{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \dots & \begin{Bmatrix} S\{h_1(s)\} \\ +\frac{a_{1(m-1)}}{q^n} \\ +\frac{a_{1(m-2)}}{q^{n-1}} \\ +\dots+\frac{a_{10}}{q^{n-m+1}} \end{Bmatrix} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \dots & \begin{Bmatrix} S\{h_2(s)\} \\ +\frac{a_{2(m-1)}}{q^n} \\ +\frac{a_{2(m-2)}}{q^{n-1}} \\ +\dots+\frac{a_{20}}{q^{n-m+1}} \end{Bmatrix} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & -S\{K_{2n}(s)\} & \dots & \begin{Bmatrix} S\{h_n(s)\} \\ +\frac{a_{n(m-1)}}{q^n} \\ +\frac{a_{n(m-2)}}{q^{n-1}} \\ +\dots+\frac{a_{n0}}{q^{n-m+1}} \end{Bmatrix} \end{vmatrix}}{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \dots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \dots & -S\{K_{2n}(s)\} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \dots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}$$

After simplification of above equations, we get the values of $S\{v_1(s)\}, S\{v_2(s)\}, \dots, S\{v_n(s)\}$. After taking the inverse ESS transform on these values, we obtain the required values of $v_1(s),\ v_2(s), \dots, v_n(s)$.

APPLICATIONS

In this part of the paper, some applications have been considered for illustrate the complete methodology.

Application 1. Consider the following (S-LVIDE-SK).

$$\left.\begin{aligned} v_1'(s) &= 1 + s - \frac{s^2}{2} + \frac{s^3}{3} + \int_0^s (s-t)v_1(t)dt + \int_0^s (s-t+1)v_2(t)dt \\ v_2'(s) &= -1 - 3s - \frac{3s^2}{2} - \frac{s^3}{3} + \int_0^s (s-t+1)v_1(t)dt + \int_0^s (s-t)v_2(t)dt \end{aligned}\right\} \tag{8}$$

With

$$v_1(0) = 1\,,\ v_2(0) = 1 \tag{9}$$

Operating SEE transform on system (8) and using convolution theorem of SEE transform, we have

$$\left.\begin{aligned} S\{v_1'(s)\} &= \begin{bmatrix} S\{1\}+S\{s\} \\ -\frac{1}{2}S\{s^2\}+\frac{1}{3}S\{s^3\} \end{bmatrix} + \begin{bmatrix} q^n S\{s\}S\{v_1(s)\} \\ +q^n S\{s+1\}S\{v_2(s)\} \end{bmatrix} \\ S\{v_2'(s)\} &= \begin{bmatrix} -S\{1\}-3S\{s\} \\ -\frac{3}{2}S\{s^2\}-\frac{1}{3}S\{s^3\} \end{bmatrix} + \begin{bmatrix} q^n S\{s+1\}S\{v_1(s)\} \\ +q^n S\{s\}S\{v_2(s)\} \end{bmatrix} \end{aligned}\right\} \tag{10}$$

Using the property “SEE transforms of derivatives” on system (10), we have

$$\left.\begin{aligned} \frac{-1}{q^n}v_1(0)+qS\{v_1(s)\} &= \begin{bmatrix} \frac{1}{q^{n+1}}+\frac{1}{q^{n+2}} \\ -\frac{1}{2}\frac{2!}{q^{n+3}}+\frac{1}{3}\frac{3!}{q^{n+4}} \end{bmatrix} + \begin{bmatrix} q^n\frac{1}{q^{n+2}}S\{v_1(s)\} \\ +q^n\left(\frac{1}{q^{n+2}}+\frac{1}{q^{n+1}}\right)S\{v_2(s)\} \end{bmatrix} \\ \frac{-1}{q^n}v_2(0)+qS\{v_2(s)\} &= \begin{bmatrix} \frac{-1}{q^{n+1}}-\frac{3}{q^{n+2}} \\ -\frac{3}{2}\frac{2!}{q^{n+3}}-\frac{1}{3}\frac{3!}{q^{n+4}} \end{bmatrix} + \begin{bmatrix} q^n\left(\frac{1}{q^{n+2}}+\frac{1}{q^{n+1}}\right)S\{v_1(s)\} \\ +q^n\frac{1}{q^{n+2}}S\{v_2(s)\} \end{bmatrix} \end{aligned}\right\} \tag{11}$$

Using equation (9) in system (11), we get

$$\left.\begin{aligned} \frac{-1}{q^n}*1+qS\{v_1(s)\} &= \begin{bmatrix} \frac{1}{q^{n+1}}+\frac{1}{q^{n+2}} \\ -\frac{1}{2}\frac{2!}{q^{n+3}}+\frac{1}{3}\frac{3!}{q^{n+4}} \end{bmatrix} + \begin{bmatrix} q^n\frac{1}{q^{n+2}}S\{v_1(s)\} \\ +q^n\left(\frac{1}{q^{n+2}}+\frac{1}{q^{n+1}}\right)S\{v_2(s)\} \end{bmatrix} \\ \frac{-1}{q^n}*1+qS\{v_2(s)\} &= \begin{bmatrix} \frac{-1}{q^{n+1}}-\frac{3}{q^{n+2}} \\ -\frac{3}{2}\frac{2!}{q^{n+3}}-\frac{1}{3}\frac{3!}{q^{n+4}} \end{bmatrix} + \begin{bmatrix} q^n\left(\frac{1}{q^{n+2}}+\frac{1}{q^{n+1}}\right)S\{v_1(s)\} \\ +q^n\frac{1}{q^{n+2}}S\{v_2(s)\} \end{bmatrix} \end{aligned}\right\} \tag{12}$$

After simplification system (12), we have

$$\left.\begin{aligned} \left(q-\frac{1}{q^2}\right)S\{v_1(s)\}-\left(\frac{1}{q^2}+\frac{1}{q}\right)S\{v_2(s)\} &= \left(\frac{1}{q^n}+\frac{1}{q^{n+1}}+\frac{1}{q^{n+2}}-\frac{1}{q^{n+3}}+\frac{2}{q^{n+4}}\right) \\ -\left(\frac{1}{q^2}+\frac{1}{q}\right)S\{v_1(s)\}+\left(q-\frac{1}{q^2}\right)S\{v_2(s)\} &= \left(\frac{1}{q^n}-\frac{1}{q^{n+1}}-\frac{3}{q^{n+2}}-\frac{3}{q^{n+3}}-\frac{2}{q^{n+4}}\right) \end{aligned}\right\} \tag{13}$$

The solution of system (13) is given by

$$\left.\begin{aligned} \left[S\{v_1(s)\} = \frac{\begin{vmatrix} \left(\frac{1}{q^n}+\frac{1}{q^{n+1}}+\frac{1}{q^{n+2}}-\frac{1}{q^{n+3}}+\frac{2}{q^{n+4}}\right) & -\left(\frac{1}{q^2}+\frac{1}{q}\right) \\ \left(\frac{1}{q^n}-\frac{1}{q^{n+1}}-\frac{3}{q^{n+2}}-\frac{3}{q^{n+3}}-\frac{2}{q^{n+4}}\right) & \left(q-\frac{1}{q^2}\right) \end{vmatrix}}{\begin{vmatrix} \left(q-\frac{1}{q^2}\right) & -\left(\frac{1}{q^2}+\frac{1}{q}\right) \\ -\left(\frac{1}{q^2}+\frac{1}{q}\right) & \left(q-\frac{1}{q^2}\right) \end{vmatrix}} = \frac{1}{q^n}\left[\frac{q^7+q^6+2q^5-2q^4-3q^3-7q^2-4q-4}{q^3(q^2+1)(q^3-q-2)}\right] \right] \\ \left[S\{v_2(s)\} = \frac{\begin{vmatrix} \left(q-\frac{1}{q^2}\right) & \left(\frac{1}{q^n}+\frac{1}{q^{n+1}}+\frac{1}{q^{n+2}}-\frac{1}{q^{n+3}}+\frac{2}{q^{n+4}}\right) \\ -\left(\frac{1}{q^2}+\frac{1}{q}\right) & \left(\frac{1}{q^n}-\frac{1}{q^{n+1}}-\frac{3}{q^{n+2}}-\frac{3}{q^{n+3}}-\frac{2}{q^{n+4}}\right) \end{vmatrix}}{\begin{vmatrix} \left(q-\frac{1}{q^2}\right) & -\left(\frac{1}{q^2}+\frac{1}{q}\right) \\ -\left(\frac{1}{q^2}+\frac{1}{q}\right) & \left(q-\frac{1}{q^2}\right) \end{vmatrix}} = \frac{1}{q^n}\left[\frac{q^7-q^6-2q^5-2q^4+q^3+3q^2+4q+4}{q^3(q^2+1)(q^3-q-2)}\right] \right] \end{aligned}\right\} \tag{14}$$

Now

$$S\{v_1(s)\} = \frac{1}{q^n}\left[\frac{q^7+q^6+2q^5-2q^4-3q^3-7q^2-4q-4}{q^3(q^2+1)(q^3-q-2)}\right] = \frac{1}{q^n}\left[\frac{A_1}{q}+\frac{B_1}{q^2}+\frac{C_1}{q^3}+\frac{D_1q+E_1}{(q^2+1)}+\frac{F_1q^2+G_1q+H_1}{(q^3-q-2)}\right]$$

$$S\{v_2(s)\} = \frac{1}{q^n}\left[\frac{q^7-q^6-2q^5-2q^4+q^3+3q^2+4q+4}{q^3(q^2+1)(q^3-q-2)}\right] = \frac{1}{q^n}\left[\frac{A_2}{q}+\frac{B_2}{q^2}+\frac{C_2}{q^3}+\frac{D_2q+E_2}{(q^2+1)}+\frac{F_2q^2+G_2q+H_2}{(q^3-q-2)}\right]$$

After simple computations, we get:

$$A_1 = B_1 = 1, C_1 = 2, D_1 = E_1 = F_1 = G_1 = H_1 = 0$$

and $\quad A_2 = 1, B_2 = -1, C_2 = -2, D_2 = E_2 = F_2 = G_2 = H_2 = 0$

Then
$$\left\{\begin{array}{l} S\{v_1(s)\} = \frac{1}{q^n}\left[\frac{1}{q} + \frac{1}{q^2} + \frac{2}{q^3}\right] = \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + \frac{2}{q^{n+3}} \\ \{v_2(s)\} = \frac{1}{q^n}\left[\frac{1}{q} - \frac{1}{q^2} - \frac{2}{q^3}\right] = \frac{1}{q^{n+1}} - \frac{1}{q^{n+2}} - \frac{2}{q^{n+3}} \end{array}\right\} \tag{15}$$

Operating inverse SEE transforms on system (15), we get the required solution of system (8) with (9) as

$$\left.\begin{array}{l} v_1(s) = S^{-1}\left\{\frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + \frac{1}{q^{n+3}}\right\} = S^{-1}\left\{\frac{1}{q^{n+1}}\right\} + S^{-1}\left\{\frac{1}{q^{n+2}}\right\} + 2S^{-1}\left\{\frac{1}{q^{n+3}}\right\} = 1 + s + s^2 \\ v_2(s) = S^{-1}\left\{\frac{1}{q^{n+1}} - \frac{1}{q^{n+2}} - \frac{2}{q^{n+3}}\right\} = S^{-1}\left\{\frac{1}{q^{n+1}}\right\} - S^{-1}\left\{\frac{1}{q^{n+2}}\right\} - 2S^{-1}\left\{\frac{1}{q^{n+3}}\right\} = 1 - s - s^2 \end{array}\right\}$$

Application 2. Consider the following (S-LVIDE-SK)

$$\left.\begin{array}{l} v_1''(s) = -s^3 - s^4 + \int_0^s 3v_2(t)dt + \int_0^s 4v_3(t)dt \\ v_2''(s) = 2 + s^2 - s^4 + \int_0^s 4v_3(t)dt - \int_0^s 2v_1(t)dt \\ v_3''(s) = 6s - s^2 + s^3 + \int_0^s 2v_1(t)dt - \int_0^s 3v_2(t)dt \end{array}\right\} \tag{16}$$

$$\text{with } v_1(0) = 0,\ v_1'(0) = 1;\ v_2(0) = 0,\ v_2'(0) = 0; v_3(0) = 0, v_3'(0) = 0 \tag{17}$$

Operating SEE transform on system (16) and using convolution theorem of SEE transform, we have

$$\left.\begin{array}{l} S\{v_1''(s)\} = -S\{s^3\} - S\{s^4\} + q^n S\{3\}S\{v_2(s)\} + q^n S\{4\}S\{v_3(s)\} \\ S\{v_2''(s)\} = S\{2\} + S\{s^2\} - S\{s^4\} + q^n S\{4\}S\{v_3(s)\} - q^n S\{2\}S\{v_1(s)\} \\ S\{v_3''(s)\} = 6S\{s\} - S\{s^2\} + S\{s^3\} + q^n S\{2\}S\{v_1(s)\} - q^n S\{3\}S\{v_2(s)\} \end{array}\right\} \tag{18}$$

Using the property "SEE transforms of derivatives" on system (18), we have

$$\left.\begin{array}{l} \frac{-v_1'(0)}{q^n} - \frac{v_1(0)}{q^{n-1}} + q^2 S\{v_1(s)\} = \begin{bmatrix} -\frac{3!}{q^{n+4}} \\ -\frac{4!}{q^{n+5}} \end{bmatrix} + \begin{bmatrix} 3q^n \frac{1}{q^{n+1}} S\{v_2(s)\} \\ +4q^n \frac{1}{q^{n+1}} S\{v_3(s)\} \end{bmatrix} \\ \frac{-v_2'(0)}{q^n} - \frac{v_2(0)}{q^{n-1}} + q^2 S\{v_2(s)\} = \begin{bmatrix} \frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} \\ -\frac{4!}{q^{n+5}} \end{bmatrix} + \begin{bmatrix} 4q^n \frac{1}{q^{n+1}} S\{v_3(s)\} \\ -2q^n \frac{1}{q^{n+1}} S\{v_1(s)\} \end{bmatrix} \\ \frac{-v_3'(0)}{q^n} - \frac{v_3(0)}{q^{n-1}} + q^2 S\{v_3(s)\} = \begin{bmatrix} \frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} \\ +\frac{3!}{q^{n+4}} \end{bmatrix} + \begin{bmatrix} 2q^n \frac{1}{q^{n+1}} S\{v_1(s)\} \\ -3q^n \frac{1}{q^{n+1}} S\{v_2(s)\} \end{bmatrix} \end{array}\right\} \tag{19}$$

Using equation (17) in system (19), we get

$$\left.\begin{array}{l}\frac{-1}{q^n}+q^2S\{v_1(s)\}=\begin{bmatrix}-\frac{3!}{q^{n+4}}\\ -\frac{4!}{q^{n+5}}\end{bmatrix}+\begin{bmatrix}3q^n\frac{1}{q^{n+1}}S\{v_2(s)\}\\ +4q^n\frac{1}{q^{n+1}}S\{v_3(s)\}\end{bmatrix}\\ q^2S\{v_2(s)\}=\begin{bmatrix}\frac{2}{q^{n+1}}+\frac{2!}{q^{n+3}}\\ -\frac{4!}{q^{n+5}}\end{bmatrix}+\begin{bmatrix}4q^n\frac{1}{q^{n+1}}S\{v_3(s)\}\\ -2q^n\frac{1}{q^{n+1}}S\{v_1(s)\}\end{bmatrix}\\ q^2S\{v_3(s)\}=\begin{bmatrix}\frac{6}{q^{n+2}}-\frac{2!}{q^{n+3}}\\ +\frac{3!}{q^{n+4}}\end{bmatrix}+\begin{bmatrix}2q^n\frac{1}{q^{n+1}}S\{v_1(s)\}\\ -3q^n\frac{1}{q^{n+1}}S\{v_2(s)\}\end{bmatrix}\end{array}\right\}\tag{20}$$

After simplification system (20), we have

$$\left.\begin{array}{l}q^2S\{v_1(s)\}-\frac{3}{q}S\{v_2(s)\}-\frac{4}{q}S\{v_3(s)\}=\left(\frac{1}{q^n}-\frac{3!}{q^{n+4}}-\frac{4!}{q^{n+5}}\right)\\ \frac{2}{q}S\{v_1(s)\}+q^2S\{v_2(s)\}-\frac{4}{q}S\{v_3(s)\}=\left(\frac{2}{q^{n+1}}+\frac{2!}{q^{n+3}}-\frac{4!}{q^{n+5}}\right)\\ \frac{-2}{q}S\{v_1(s)\}+\frac{3}{q}S\{v_2(s)\}+q^2S\{v_3(s)\}=\left(\frac{6}{q^{n+2}}-\frac{2!}{q^{n+3}}+\frac{3!}{q^{n+4}}\right)\end{array}\right\}\tag{21}$$

The solution of system (21) is given by

$$S\{v_1(s)\}=\frac{\begin{vmatrix}(\frac{1}{q^n}-\frac{3!}{q^{n+4}}-\frac{4!}{q^{n+5}}) & -\frac{3}{q} & -\frac{4}{q}\\ (\frac{2}{q^{n+1}}+\frac{2!}{q^{n+3}}-\frac{4!}{q^{n+5}}) & q^2 & -\frac{4}{q}\\ (\frac{6}{q^{n+2}}-\frac{2!}{q^{n+3}}+\frac{3!}{q^{n+4}}) & \frac{3}{q} & q^2\end{vmatrix}}{\begin{vmatrix}q^2 & -\frac{3}{q} & -\frac{4}{q}\\ \frac{2}{q} & q^2 & -\frac{4}{q}\\ \frac{-2}{q} & \frac{3}{q} & q^2\end{vmatrix}}=\frac{\frac{1}{q^n}[q^4+\frac{10}{q^2}-\frac{48}{q^5}]}{q^6+10-\frac{48}{q^3}}=\frac{\frac{1}{q^n}*\frac{1}{q^2}[q^6+10-\frac{48}{q^3}]}{[q^6+10-\frac{48}{q^3}]}=\frac{1}{q^{n+2}}\tag{22}$$

$$S\{v_2(s)\}=\frac{\begin{vmatrix}q^2 & (\frac{1}{q^n}-\frac{3!}{q^{n+4}}-\frac{4!}{q^{n+5}}) & -\frac{4}{q}\\ \frac{2}{q} & (\frac{2}{q^{n+1}}+\frac{2!}{q^{n+3}}-\frac{4!}{q^{n+5}}) & -\frac{4}{q}\\ \frac{-2}{q} & (\frac{6}{q^{n+2}}-\frac{2!}{q^{n+3}}+\frac{3!}{q^{n+4}}) & q^2\end{vmatrix}}{\begin{vmatrix}q^2 & -\frac{3}{q} & -\frac{4}{q}\\ \frac{2}{q} & q^2 & -\frac{4}{q}\\ \frac{-2}{q} & \frac{3}{q} & q^2\end{vmatrix}}=\frac{\frac{1}{q^n}[2q^3+\frac{20}{q^3}-\frac{96}{q^6}]}{q^6+10-\frac{48}{q^3}}=\frac{\frac{2}{q^3}[q^6+10-\frac{48}{q^3}]}{q^n[q^6+10-\frac{48}{q^3}]}=\frac{2!}{q^{n+3}}\tag{23}$$

$$S\{v_3(s)\}=\frac{\begin{vmatrix}q^2 & -\frac{3}{q} & (\frac{1}{q^n}-\frac{3!}{q^{n+4}}-\frac{4!}{q^{n+5}})\\ \frac{2}{q} & q^2 & (\frac{2}{q^{n+1}}+\frac{2!}{q^{n+3}}-\frac{4!}{q^{n+5}})\\ \frac{-2}{q} & \frac{3}{q} & (\frac{6}{q^{n+2}}-\frac{2!}{q^{n+3}}+\frac{3!}{q^{n+4}})\end{vmatrix}}{\begin{vmatrix}q^2 & -\frac{3}{q} & -\frac{4}{q}\\ \frac{2}{q} & q^2 & -\frac{4}{q}\\ \frac{-2}{q} & \frac{3}{q} & q^2\end{vmatrix}}=\frac{\frac{1}{q^n}[6q^2+\frac{60}{q^4}-\frac{288}{q^7}]}{q^6+10-\frac{48}{q^3}}=\frac{\frac{6}{q^4}[q^6+10-\frac{48}{q^3}]}{q^n[q^6+10-\frac{48}{q^3}]}=\frac{3!}{q^{n+4}}\tag{24}$$

Taking inverse SEE transforms on equations (22), (23), and (24) we obtain the required solution of the system (16) with (17) as

$$v_1(s)=S^{-1}\left\{\frac{1}{q^{n+2}}\right\}=S$$

$$v_2(s) = S^{-1}\left\{\frac{2!}{q^{n+3}}\right\} = s^2$$

$$v_3(s) = S^{-1}\left\{\frac{3!}{q^{n+4}}\right\} = s^3$$

CONCLUSION

In this paper, we successfully obtained the solution of (S-LVIDE-SK) by SEE transform and the methodology completely illustrated by giving two applications. The results of these applications assert that the SEE transform is a very effective and beneficial integral transform for obtaining the exact solution of (S-LVIDE-SK). The proposed scheme can be applied to system of linear Volterra-integral equations.

APPENDICES

TABLE 1. USEFULL PROPERTIES OF SEE TRANSFORM Mansou et al. (2021), Ajel Mansour et al. (2021)

S.N	Name of Property	Mathematical Form
1.	Linearity	$S\{cW_1(t) + dW_2(t)\} = cS\{W_1(t)\} + dS\{W_2(t)\}$
2.	Change of Scale	$S\{W(ct)\} = \frac{1}{c^{n+1}} G(\frac{q}{c})$
3.	Shifting	$S\{e^{ct}W(t)\} = \frac{(q-c)^n}{q^n} G(q-c)$
		$S\{tW(t)\} = \{-\frac{n}{q} - \frac{d}{dq}\} G(q)$
4.	1st Derivative	$S\{W'(t)\} = -\frac{1}{q^n} W(0) + qG(q)$
5.	2nd Derivative	$S\{W''(t)\} = -\frac{W'(0)}{q^n} - \frac{W(0)}{q^{n-1}} + q^2G(q)$
6.	mth Derivative	$S\{W^{(m)}(t)\} = -\frac{W^{(m-1)}(0)}{q^n} - \frac{W^{(m-2)}(0)}{q^{n-1}} \ldots\ldots - \frac{W(0)}{q^{n-m+1}} + q^m G(q)$
7.	Convolution	$S\{W_1(t) * W_2(t)\} = q^n S\{W_1(t)\} * S\{W_2(t)\}$

TABLE 2. SEE TRANSFORM OF USEFUL FUNCTIONS **Ajel Mansour et al. (2021)**

S.N	W(t)	S{W(t)} = G(q)
1.	1	$\frac{1}{q^{n+1}}$
2.	t	$\frac{1}{q^{n+2}}$
3.	t^2	$\frac{2!}{q^{n+3}}$
4.	$t^m; m \in Z^+$	$\frac{m!}{q^{n+m+1}}$
5.	e^{ct}	$\frac{1}{q^n(q-c)}$
6.	sinct	$\frac{c}{q^n(q^2+c^2)}$
7.	cosc$t$	$\frac{1}{q^{n-1}(q^2+c^2)}$
8.	sinhct	$\frac{c}{q^n(q^2-c^2)}$
9.	coshct	$\frac{1}{q^{n-1}(q^2-c^2)}$

TABLE 3. INVERS SEE TRANSFORM OF USEFUL FUNCTIONS Ajel Mansour et al. (2021)

S.N	G(q)	$W(t) = K^{-1}\{G(q)\}$
1.	$\frac{1}{q^{n+1}}$	1
2.	$\frac{1}{q^{n+2}}$	T
3.	$\frac{2!}{q^{n+3}}$	t^2
4.	$\frac{m!}{q^{n+m+1}}; m, n \in Z^+$	t^m
5.	$\frac{1}{q^n(q-c)}$	e^{ct}
6.	$\frac{c}{q^n(q^2+c^2)}$	Sinct
7.	$\frac{1}{q^{n-1}(q^2+c^2)}$	Cosct
8.	$\frac{c}{q^n(q^2-c^2)}$	Sinhct
9.	$\frac{1}{q^{n-1}(q^2-c^2)}$	Coshct

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